

# Examples of Gorenstein domains and symbolic powers of monomial space curves

Peter Schenzel

*Sektion Mathematik, Martin-Luther-Universität Halle–Wittenberg O-4010, Halle,  
Federal Republic of Germany*

Communicated by J.D. Sally

Received 29 September 1989

Revised 20 February 1990

Dedicated to Professor Hideyuki Matsumura on his sixtieth birthday

## 1. Introduction

Monomial curves  $C$  in  $\mathbb{P}_k^3$ ,  $k$  an arbitrary field, are those curves whose generic zero is given by

$$x = s^d, \quad y = s^{d-a}t^a, \quad z = s^{d-b}t^b, \quad w = t^d, \quad 1 \leq a, b < d,$$

with  $\gcd(a, b, d) = 1$ . These curves are a good source for testing several questions in algebraic geometry and commutative algebra. Let  $P$  denote the defining prime ideal of  $C$  in  $A = k[x, y, z, w]$ . In the case  $C$  is smooth, which holds if and only if  $a = d - b = 1$ , Brumatti, Simis and Vasconcelos [5] have shown that  $P^{(n)} = \overline{P^n}$  for all  $n \geq 1$ . Here  $P^{(n)}$  respectively  $\overline{P^n}$  denotes the  $n$ th symbolic power of  $P$  respectively the integral closure of  $P^n$ . For  $d \leq 5$  they verified the normality of  $R(P)$ , the Rees ring of  $A$  with respect to  $P$ , i.e.,  $P^{(n)} = \overline{P^n} = P^n$  for all  $n \geq 1$ . In the case  $d = 4$ , this was shown independently by Trung respectively by Eisenbud and Huneke, see [20] respectively [7]. One of the main objects of the present paper is to prove the following theorem:

**Theorem 1.**<sup>1</sup> *Let  $P$  denote the prime ideal of the monomial curve  $C \subseteq \mathbb{P}_k^3$  given parametrically by*

<sup>1</sup> In the preprint “Powers of ideals having small analytic deviation”, S. Huckaba and C. Huneke gave a different proof of this result.

$$x = s^{a+b}, \quad y = s^b t^a, \quad z = s^a t^b, \quad w = t^{a+b}$$

with integers  $1 \leq a < b$  and  $\gcd(a, b) = 1$ . Then it holds:

- (1) The Rees ring  $R(P) = \bigoplus_{n \geq 0} P^n t^n$  is a normal Gorenstein domain.
- (2) The form ring  $G(P) = \bigoplus_{n \geq 0} P^n / P^{n+1}$  is a Gorenstein domain.

By [4], it is known that  $P$  is arithmetically Cohen–Macaulay respectively Buchsbaum if and only if  $b - a \leq 1$  respectively  $b - a \leq 2$ . In particular, by view of Hochster’s results (see [9]), it turns out that  $P^n = P^{(n)} = \overline{P^n}$  for all  $n \geq 1$ . In the special case of  $a = 1$ , i.e.,  $C$  is smooth, this answers affirmatively a question posed in [5]. Theorem 1 is proved in Section 3. To this end there is also a presentation of  $R(P)$  respectively  $G(P)$  as a quotient of a certain polynomial ring. By virtue of Theorem 1 one may ask whether this holds for all monomial curves in  $\mathbb{P}_k^3$ . This is not true (see [15]), if the semigroup generated by  $a, b, d$  is not symmetric. Under this assumption,  $C$  does not have a complete intersection point in  $(1:0:0:0)$ . Therefore,  $P^{(n)} \neq P^n$  for all  $n \geq 2$  as follows by a localization argument. In order to get a complete picture on the symbolic powers one wants to decide whether  $S(P) = \bigoplus_{n \geq 0} P^{(n)} t^n$ , the symbolic Rees ring of  $P$ , is an  $A$ -algebra of finite type. Recently this has attained much attention for the case of monomial curves in  $\mathbb{A}_k^3$ ; see, e.g., [8] and [15]. Here we are able to present the first nontrivial examples of two-dimensional monomial prime ideals  $P$  such that  $S(P)$  is an algebra of finite type. To this end consider the following two cases:

- (A)  $x = s^{a(a+b)+b}, \quad y = s^{a(a+b)-2a-b+1} t^{2(a+b)-1},$   
 $z = s^{a(a+b)-a-b} t^{a+2b}, \quad w = t^{a(a+b)+b}, \quad \text{with } 1 < a < b.$
- (B)  $x = s^{ab+2a+b-1}, \quad y = s^{ab+2a-b-2} t^{2b+1},$   
 $z = s^{ab-a-b+1} t^{3a+2b-2}, \quad w = t^{ab+2a+b-1}, \quad \text{with } 1 < a, b.$

**Theorem 2.** Let  $P$  denote the defining prime ideal of the curve given in (A) respectively (B). Then  $P$  is a perfect respectively imperfect prime ideal with  $S(P) = A[Pt, P^{(2)}t^2]$  a Gorenstein domain.

Thus, the property whether  $S(P)$  is an  $A$ -algebra of finite type does not depend upon the perfectness of  $P$ . Moreover, it turns out that  $P^{(2)} = (P^2, \Delta)$  for a certain element  $\Delta$ . Theorem 2 is proved in Section 4. Section 2 contains an auxiliary result on the Cohen–Macaulayness of an affine semigroup ring necessary for our proof of Theorem 1. In Section 5 we conclude with some remarks. Among others we relate the finiteness of the symbolic Rees ring  $S(P)$  to the presentation of  $C$  as a set-theoretic complete intersection.

## 2. An auxiliary result

In order to prove one of the main results we need the Cohen–Macaulayness of a certain affine semigroup ring. As above let  $k$  denote an arbitrary field. For two

integers  $1 \leq a < b$  with  $\gcd(a, b) = 1$  put

$$B = k[s^{a+b}, s^b t^a, s^a t^b, t^{a+b}, s^{b(b-a-i)} t^{bi} u, \quad 0 \leq i \leq b-a],$$

the subring of the polynomial ring  $k[s, t, u]$ . Let

$$R = k[x, y, z, w, T_0, \dots, T_{b-a}]$$

denote the polynomial ring in the indeterminates  $x, \dots, w, T_0, \dots, T_{b-a}$ . Then there is a natural epimorphism  $f: R \rightarrow B$  induced by

$$\begin{aligned} x &\mapsto s^{a+b}, \quad y \mapsto s^b t^a, \quad z \mapsto s^a t^b, \quad w \mapsto t^{a+b}, \\ T_i &\mapsto s^{b(b-a-i)} t^{bi} u, \quad 0 \leq i \leq b-a. \end{aligned}$$

Put  $Q = yz - xw$  and  $F_i = y^{b-i} w^i - x^{b-a-i} z^{a+i}$ ,  $0 \leq i \leq b-a$ . Then it is known (see [4]), that  $p = (Q, F_0, \dots, F_{b-a})$  is the defining prime ideal of the monomial curve with generic zero  $(s^{a+b}, s^b t^a, s^a t^b, t^{a+b})$  in  $\mathbb{P}_k^3$ . Set  $I = (p, I_2)$ , where  $I_2$  is the ideal generated by the  $2 \times 2$  minors of the following matrix:

$$\begin{pmatrix} x & y & T_0 & T_1 & \cdots & T_{b-a-1} \\ z & w & T_1 & T_2 & \cdots & T_{b-a} \end{pmatrix},$$

i.e.,  $I_2$  is generated by

$$\begin{aligned} Q, \quad wT_i - yT_{i+1}, \quad zT_i - xT_{i+1}, \quad 0 \leq i \leq b-a-1, \\ T_i T_{j+1} - T_{i+1} T_j, \quad 0 \leq i < j \leq b-a-1. \end{aligned}$$

Now we shall prove that  $I$  is equal to the kernel of the natural epimorphism  $f$ .

**Theorem 3.** *We have  $I = \ker f$ , i.e.,  $R/I \cong B$ , and  $R/I$  is a three-dimensional Cohen–Macaulay domain.*

**Proof.** It requires several steps. First note that  $I \subseteq \ker f$ , which follows by an easy computation. Therefore,  $f$  induces an epimorphism  $R/I \rightarrow B$ . In order to show that  $R/I$  is a Cohen–Macaulay ring it is enough to prove that  $T := R_M / IR_M$ ,  $M = (x, y, z, w, T_0, \dots, T_{b-a})$ , is a Cohen–Macaulay ring (see [12]). Recall that  $I$  is a homogeneous ideal of  $R$ . Let  $N$  denote the irrelevant maximal ideal of  $B$ . Then there is an epimorphism  $T \rightarrow B_N$  which shows  $\dim T \geq 3$  since  $\dim B_N = 3$ . Now we claim that  $\underline{x} = \{T_{b-a}, x, T_0 - w\}$  forms a system of parameters of  $T$  and  $\dim T = 3$ . To this end note that

$$R/(I, \underline{x}R) \cong k[y, z, w, T_1, \dots, T_{b-a-1}]/I',$$

where

$$I' = (yz, y^{b-i}w^i, 0 \leq i \leq b-a-1, y^a w^{b-a} - z^b, I'_2).$$

Here  $I'_2$  denotes the ideal generated by the  $2 \times 2$  minors of the matrix

$$\begin{pmatrix} 0 & y & w & T_1 & \cdots & T_{b-a-1} \\ z & w & T_1 & T_2 & \cdots & 0 \end{pmatrix}.$$

Then  $\text{Rad } I' = (T_{b-a-1}, \dots, T_1, w, y, z)R$  as is easily seen, i.e.,  $R/(I, \underline{x}R)$  is of finite length. Therefore,  $\dim T = 3$  and  $\underline{x}$  forms a system of parameters of  $T$ .

Next we claim that  $\underline{x}$  forms a reducing system of parameters of  $T$  in the sense of Auslander and Buchsbaum (see [1]). To this end consider several localizations. First note that

$$(R/I)_x \cong k[x, 1/x, y, z, T_0]/(F_0).$$

This follows because of

$$w = (1/x)yz, \quad F_i = (z/x)^i F_0, \quad T_{i+1} = (z/x)T_i,$$

in the case  $x$  becomes a unit. Similarly, it yields

$$(R/I)_w \cong k[y, z, w, 1/w, T_{b-a}]/(F_{b-a}),$$

$$(R/I)_{T_0} \cong k[x, y, T_0, 1/T_0, T_1]/(F_0)$$

and

$$(R/I)_{T_{b-a}} \cong k[x, w, T_{b-a-1}, T_{b-a}, 1/T_{b-a}]/(F_{b-a}).$$

Since  $\dim R_M/(I, x, w, T_0, T_{b-a})R_M = 0$  it turns out that the non-Cohen-Macaulay locus of  $T$  is contained in  $V(MT)$ .

Moreover, it shows that  $IR_M$  is unmixed up to an  $MR_M$ -primary component. By virtue of [17] every system of parameters is reducing. In order to show that  $T$  is a Cohen-Macaulay ring it is enough to prove that  $\underline{x}$  forms a  $T$ -regular sequence. Because it is a reducing system of parameters it is enough to prove that  $J := (I, T_{b-a}, x)R$  does not have an irrelevant primary component (see [1]). To this end first note that

$$J = (x, T_{b-a}, yz, y^{b-i}w^i, 0 \leq i \leq b-a-1, y^a w^{b-a} - z^b, J'),$$

where  $J'$  denotes the ideal generated by the  $2 \times 2$  minors of the following matrix:

$$\begin{pmatrix} 0 & y & T_0 & \cdots & T_{b-a-2} & T_{b-a-1} \\ z & w & T_1 & \cdots & T_{b-a-1} & 0 \end{pmatrix}.$$

Therefore,

$$\text{Rad } J = (T_{b-a}, \dots, T_0, y, z, x)R \cap (T_{b-a}, \dots, T_1, w, y, z, x)R.$$

Next we compute

$$J : \langle w \rangle = (x, yz, y^{a+1}, y^a w^{b-a} - z^b, T_0, \dots, T_{b-a})$$

and

$$J : \langle T_0 \rangle = (x, z, y^{b-i} w^i, 0 \leq i \leq b-a-1, w^{b-a}, J'', T_{b-a}),$$

where  $J''$  is the ideal generated by the  $2 \times 2$  minors of the matrix

$$\begin{pmatrix} y & T_0 & \cdots & T_{b-a-2} & T_{b-a-1} \\ w & T_1 & \cdots & T_{b-a-1} & 0 \end{pmatrix}.$$

Here recall that  $J : \langle u \rangle = \bigcup_{n \geq 1} J : u^n$  for an element  $u$  of  $R$ . The first equality is easy to derive. For the second note that  $I$  contains all elements of the form  $w^i T_0 - y^i T_i$ ,  $1 \leq i \leq b-a$ . Using the modular law for intersections of ideals it yields

$$\begin{aligned} J : \langle x \rangle \cap J : \langle T_0 \rangle &= (x, y^{b-i} w^i, 0 \leq i \leq b-a-1, J'', T_{b-a}, yz, y^a w^{b-a} - z^b, \\ &\quad (z, w^{b-a}) \cap (x, y^{a+1}, T_0, \dots, T_{b-a})) \\ &= J, \end{aligned}$$

because  $w^{b-a} T_i \in J''$ ,  $0 \leq i \leq b-a-1$ . Note that  $J''$  contains all elements of the form  $w^i T_j - y^i T_{j+1}$ ,  $0 \leq j < i$ ,  $j+i \leq b-a$ .

This finishes the proof of the Cohen–Macaulayness of  $R/I$ . Now  $x$  is an  $R/I$ -regular element, i.e., there is an injection

$$0 \rightarrow R/I \rightarrow (R/I)_x.$$

Because  $(R/I)_x$  is a domain the same holds for  $R/I$ . Finally,  $I = \ker f$  because both are prime ideals of the same height. Whence the proof of Theorem 3 is complete.  $\square$

By the symmetry of  $R/I$  we get also that  $\underline{x} = \{T_0, w, T_{b-a} - x\}$  forms an  $R/I$ -regular sequence. In relation to our main result Theorem 1 it will follow that  $B \cong R/I$  is in fact a Gorenstein domain; see Corollary 7.

### 3. Proof of Theorem 1

At the beginning let us fix some notation. Let  $k$  denote a field with  $s, t$  indeterminates over  $k$ . Write

$$x = s^{a+b}, \quad y = s^b t^a, \quad z = s^a t^b, \quad w = t^{a+b}$$

with integers  $1 \leq a < b$  and  $\gcd(a, b) = 1$ . By [3], the defining prime ideal  $P$  of this monomial curve in  $R = k[x, y, z, w]$  is given by  $P = (Q, F_0, \dots, F_{b-a})$ , where  $Q = yz - xw$  and  $F_i = y^{b-i} w^i - x^{b-a-i} z^{a+i}$  for  $0 \leq i \leq b-a$ . Also

$$wF_i - yF_{i+1} - x^{b-a-i-1} z^{a+i} Q = 0$$

and

$$zF_i - xF_{i+1} - y^{b-i-1} w^i Q = 0$$

for  $0 \leq i \leq b-a-1$ . These relations form a minimal generating set of the first module of syzygies of  $P$ . On the other hand, there are the following quadratic relations:

$$F_i F_{j+1} - F_{i+1} F_j + c_{ij} Q^2 = 0, \quad 0 \leq i < j \leq b-a-1,$$

where

$$c_{ij} = x^{b-a-j-1} y^{b-j-1} z^{a+i} w^i (y^{j-i} z^{j-i} - x^{j-i} w^{j-i}) / (yz - xw).$$

Note that  $c_{ij} \in M$ ,  $M = (x, y, z, w)R$ . This is easy to check.

Now let  $G(P) = \bigoplus_{n \geq 0} P^n / P^{n+1}$  denote the form ring of  $R$  with respect to  $P$ . Let

$$g: (R/P)[T_0, \dots, T_{b-a}, S] \rightarrow G(P)$$

be the natural epimorphism induced by

$$T_i \mapsto F_i, \quad 0 \leq i \leq b-a-1, \quad S \mapsto Q,$$

where  $T_0, \dots, T_{b-a}, S$  denote indeterminates over  $R$ . Let  $J$  denote the ideal of  $(R/P)[T_0, \dots, T_{b-a}, S]$  generated by

$$\begin{aligned} & wT_i - yT_{i+1} - x^{b-a-i-1} z^{a+i} S, \\ (*) \quad & zT_i - xT_{i+1} - y^{b-i-1} w^i S, \quad 0 \leq i \leq b-a-1, \\ & T_i T_{j+1} - T_{i+1} T_j + c_{ij} S^2, \quad 0 \leq i < j \leq b-a-1. \end{aligned}$$

Then  $J \subseteq \ker g$  as noted above.

**Proposition 4.** *With the previous notation we have:*

- (a)  $G(P) \cong (R/P)[T_0, \dots, T_{b-a}, S]/J$  and
- (b)  $G(P)$  is a Cohen–Macaulay domain.

**Proof.** First note that  $g$  induces an epimorphism  $\bar{g}: C \rightarrow G(P)$ , where  $C = (R/P)[T_0, \dots, T_{b-a}, S]/J$ . Modulo  $S$  it induces an epimorphism

$$\bar{\bar{g}}: C/(S) \rightarrow G(P)/(Q'),$$

where  $Q'$  denotes the initial form of  $Q$  in  $G(P)$ . With the notation of Section 2 it follows that  $C/(S) \cong R/I \cong B$ . That is,  $C/(S)$  is a Cohen–Macaulay domain with  $\dim C/(S) = 3$  and  $\dim C \geq 4$  because of  $\dim G(P) = 4$ . Therefore,  $\dim C = 4$  and  $S$  is a prime element in  $C$ . Next, we claim that  $C$  is a Cohen–Macaulay domain. To this end let  $q \subseteq (S)$  denote a minimal (homogeneous) prime ideal of  $C$ . Then

$$Sq = (S) \cap q = q$$

because  $S \notin q$ , i.e.,  $q = (0)$  and  $C$  is a domain. Because  $S$  is a  $C$ -regular element and  $C/(S)$  is a Cohen–Macaulay ring we know that  $C$  is a Cohen–Macaulay graded ring. By virtue of  $\bar{g}$  the form ring  $G(P)$  is an epimorphic image of the domain  $C$ . Now  $\dim C = \dim G(P) = 4$ , i.e.,  $\ker \bar{g} = (0)$ . This proves the statements (a) and (b).  $\square$

**Corollary 5.** *Let  $P$  denote the above-defined prime ideal. Then  $P^n = \overline{P^n} = P^{(n)}$  for all  $n \geq 1$ .*

**Proof.** By Proposition 4,  $G(P)$  is a domain with  $R_P$  a regular local ring. Then  $P^n = P^{(n)}$  for all  $n \geq 1$  by [9, Theorem 1]. Moreover,  $P^n = \overline{P^n}$  for all  $n \geq 1$  by [2].  $\square$

Note that  $P^{(n)} = \overline{P^n}$  follows also by virtue of [5, (2.14)], because  $P$  is a complete intersection in codimension one with the analytic spread  $a(P) = 3 < \dim R = 4$ . This is true because

$$(**) \quad G(P)/(M) \cong k[T_0, \dots, T_{b-a}, S]/I^*,$$

where  $I$  is generated by the  $2 \times 2$  minors of the matrix

$$\begin{pmatrix} T_0 & T_1 & \cdots & T_{b-a-1} \\ T_1 & T_2 & \cdots & T_{b-a} \end{pmatrix}.$$

In order to continue with the proof of Theorem 1, let  $J$  be the ideal of  $R[T_0, \dots, T_{b-a}, S]$  generated by the relations (\*). Consider  $R(P) = R[Pt] = \bigoplus_{n \geq 0} P^n t^n$ , the Rees algebra of  $R$  with respect to  $P$ .

**Proposition 6.** *With the above notation we have:*

- (a)  $R(P) \cong R[T_0, \dots, T_{b-a}, S]/J$  and
- (b)  $R(P)$  is a Gorenstein domain.

**Proof.** First note that there is a natural epimorphism

$$h: R[T_0, \dots, T_{b-a}, S] \rightarrow R(P)$$

induced by  $T_i \mapsto F_i t$ ,  $0 \leq i \leq b-a$ ,  $S \mapsto Qt$ . Clearly  $J \subseteq \ker h$ . By (\*\*) it follows that  $\underline{F} = \{F_0, F_{b-a}, Q\}$  forms a minimal reduction with reduction exponent  $r_{\underline{F}}(P) = 1$ , i.e.,  $P^2 = \underline{F}P$ . Now  $P$  is generically a complete intersection with grade  $G(P)_+ = 2$ ; see the proof of Proposition 4. Hence it holds that  $n(P) = 2$  for the relation type  $n(P)$  of  $P$ ; see [10, Corollary 1.6] or [15, (4.2)]. Therefore,  $\ker h$  is minimally generated by forms of degree  $\leq 2$ . By Proposition 4 it follows that  $J = \ker h$ , i.e., claim (a) is true. In order to prove (b), first note that  $S$  is a regular element with respect to

$$C = R[T_0, \dots, T_{b-a}, S]/J.$$

Now consider  $C/(S)$ . By a localization argument looking at  $(C/(S))_u$ ,  $u = x, y, z, w, T_0, T_{b-a}$ , it turns out that the non-Cohen–Macaulay locus of  $C/(S)$  is at most zero-dimensional. Therefore, the non-Cohen–Macaulay locus of  $C$  is at most one-dimensional. On the other hand,  $R(P) \cong C$  is a quasi-Gorenstein domain by virtue of [18, Corollary 3.4]. It is enough to show that  $R(P)_N$ ,  $N = R(P)_+$ , is a Cohen–Macaulay ring, see [12]. Because of depth  $G(P) = 4$ , we have that

$$4 \leq \text{depth } R_N \leq \dim R_N/p + \text{depth } (R_N)_p \leq 5$$

for all prime ideals  $p$  of  $R_N$  and  $R = R(P)$ . In the sense of [14, 3.2.7],  $C$  is half-way Cohen–Macaulay and thus a Cohen–Macaulay ring. Because quasi-Gorenstein and Cohen–Macaulay means Gorenstein the claim follows.  $\square$

**Corollary 7.** *With the above notation,  $G(P)$  respectively the ring  $B$  of Theorem 3 is a Gorenstein ring.*

**Proof.** Because  $R(P)$  is a Gorenstein domain, it follows that  $G(P)$  is a Gorenstein ring (see [11]). Since  $G(P)/(Q') \cong B$  it yields also that  $B$  is a Gorenstein ring because  $Q'$  is a  $G(P)$ -regular element.  $\square$

#### 4. Proof of Theorem 2

We start with the investigations on the first monomial curve. Fix the generic point of the monomial curve in  $\mathbb{P}_k^3$  given in (A) of Section 1. It turns out (see [3]),



that the defining prime ideal  $P$  in  $A = k[x, y, z, w]$  is given by  $P = (F, G, H)$ , where

$$F = z^{a+b} - x^{a-1}y^bw, \quad G = x^{a-1}w^2 - y^az, \quad H = y^{a+b} - z^{a+b-1}w.$$

Therefore,  $P$  is generated by the  $2 \times 2$  minors of the matrix

$$\begin{pmatrix} y^a & z^{a+b-1} & x^{a-1}w \\ w & y^b & z \end{pmatrix}.$$

That is,  $P$  is a perfect prime ideal of height two. Taking the affine piece given by  $x = 1$  and using the main results of [15] it follows that

$$\Delta = z^{a+b-1}F + x^{a-1}y^{b-1}z^{a+b-2}wG + x^{a-1}y^bH$$

satisfies the following relations:

$$y^a\Delta = x^{a-1}H^2 - z^{a+b-2}FG,$$

$$z\Delta = F^2 - x^{a-1}y^{b-a}GH,$$

$$w\Delta = y^{b-a}z^{a+b-2}G^2 - FH.$$

We have  $\Delta \in P^{(2)} \setminus P^2$ . Now let  $S(P) = \bigoplus_{n \geq 0} P^{(n)}t^n$  denote the symbolic Rees ring of  $A$  with respect to  $P$ .

**Proposition 8.** *With the previous notation we have:*

- (a)  $S(P) = A[Pt, \Delta t^2]$ , and
- (b)  $S(P)$  is a Gorenstein domain.

**Proof.** Set  $R = k[x, y, z, w, U, V, W, S]$ . Then there is the following epimorphism  $f: R \rightarrow A[Pt, \Delta t^2]$  induced by  $U \mapsto Ft$ ,  $V \mapsto Gt$ ,  $W \mapsto Ht$ ,  $S \mapsto \Delta t^2$ . Then  $I \subseteq \ker f$ , where  $I$  denotes the ideal of  $R$  generated by

$$y^aU + z^{a+b-1}V + x^{a-1}wW, \quad wU + y^bV + zW,$$

$$y^aS - x^{a-1}W^2 + z^{a+b-2}UV,$$

$$zS - U^2 + x^{a-1}y^{b-a}WV, \quad wS - y^{b-a}z^{a+b-2}V^2 + UW.$$

Thus  $f$  induces an epimorphism  $\bar{f}: R/I \rightarrow A[Pt, \Delta^2]$ , i.e.,  $\dim R/I \geq 5$ . On the other hand, the ideal  $I$  is generated by the Pfaffians of the following skew symmetric matrix:

$$\begin{pmatrix} 0 & z^{a+b-2}V & U & x^{a-1}W & -S \\ -z^{a+b-2}V & 0 & w & -y^a & W \\ -U & -w & 0 & z & y^{b-a}V \\ -x^{a-1}W & y^a & -z & 0 & U \\ S & -W & -y^{b-a}V & -U & 0 \end{pmatrix}.$$

Now consider the system of elements  $\underline{x} = \{z, S, W - x, V - y, U - w\}$ . Then  $\text{Rad}(I, \underline{x}R) = (x, y, z, w, U, V, W, S)R$  as is easily seen. Therefore,  $\dim R/I = 5$  and  $I$  is an ideal of height 3. By the Buchsbaum–Eisenbud Structure Theorem (see [6]), it turns out that  $R/I$  is a five-dimensional Gorenstein ring with  $\underline{x}$  an  $R/I$ -regular sequence. An easy computation yields

$$(R/I)_z \cong k[x, y, z, 1/z, w, U, V]/(GU - FV).$$

From the injection  $R/I \rightarrow (R/I)_z$  it follows that  $R/I$  is a domain with  $R/I \cong A[Pt, \Delta t^2]$  by an argument on the dimensions. Because  $(1:0:0:0)$  is the only noncomplete intersection point of  $P$  we see that  $S(P) = T_{pR(P)}(R(P))$ , where  $T_{pR(P)}(R(P))$  denotes the ideal transform of  $R(P)$  with respect to  $pR(P)$ ,  $p = (y, z, w)$ . Now  $R/I$  is a Gorenstein ring with  $\text{ht } p(R/I) = 2$ , i.e.,  $T_{p(R/I)}(R/I) = R/I$ . Now the inclusions  $R(P) \subseteq R/I \subseteq S(P)$  imply  $S(P) \cong R/I$ , which proves (a) and (b).  $\square$

In order to prove the second part of Theorem 2, fix the generic point of the monomial curve given in (B) of Section 1. Then the defining prime ideal  $P$  in  $A = k[x, y, z, w]$  is given by  $P = (F_0, F_1, F_2, F_3)$  with

$$\begin{aligned} F_0 &= y^a z - x^{a-1} w^2, & F_1 &= y^{2a+b-1} - x^{2a-2} z^b w, \\ F_2 &= y^{a+b-1} w - x^{a-1} z^{b+1}, & F_3 &= y^{b-1} w^3 - z^{b+2} \end{aligned}$$

(see [3]). In particular,  $P$  is not a perfect prime ideal. Moreover, note that  $P$  is generated by the  $2 \times 2$  minors of the matrix

$$\begin{pmatrix} y^a & x^{a-1} z^b & w & z^{b+1} \\ x^{a-1} w & y^{a+b-1} & z & w^2 y^{b-1} \end{pmatrix}.$$

Taking the affine piece of the curve at  $x = 1$  and using the main results of [15] it follows that

$$\begin{aligned} y^a \Delta &= F_1^2 - x^{2a-2} z^{b-1} F_0 F_2, \\ z \Delta &= x^{a-1} F_2^2 + y^{b-1} F_0 F_1, \\ w \Delta &= x^{a-1} y^{b-1} z^{b-1} F_0^2 + F_1 F_2, \end{aligned}$$

where

$$\Delta = -x^{2a-2}z^bF_2 - x^{2a-2}y^{b-1}z^{b-1}wF_0 + y^{a+b-1}F_1.$$

Therefore, we get that  $\Delta \in P^{(2)} \setminus P^2$ .

**Proposition 9.** *With the previous notation we have:*

- (a)  $S(P) = A[Pt, \Delta t^2]$ , and
- (b)  $S(P)$  is a Gorenstein domain.

**Proof.** Let  $R = k[x, y, z, w, T_0, T_1, T_2, T_3, S]$ . Then there is an epimorphism  $f: R \rightarrow A[Pt, \Delta t^2]$  induced by  $T_i \mapsto F_i t$ ,  $i = 0, \dots, 3$ ,  $S \mapsto \Delta t^2$ . Let  $I$  denote the ideal generated by the following forms:

$$\begin{aligned} x^{a-1}z^bT_0 - wT_1 + y^aT_2, & \quad y^{a+b-1}T_0 - zT_1 + x^{a-1}wT_2, \\ z^{b+1}T_0 - w^2T_2 + y^aT_3, & \quad y^{b-1}wT_0 - zT_2 + x^{a-1}T_3, \\ y^{b-1}z^bT_0^2 + T_1T_3 - wT_2^2, & \quad y^aS - T_1^2 + x^{2a-2}z^{b-1}T_0T_2, \\ zS - x^{a-1}T_2^2 - y^{b-1}T_0T_1, & \quad wS - x^{a-1}y^{b-1}z^{b-1}T_0^2 - T_1T_2, \\ T_2^3 - T_3S - y^{2b-2}z^{b-1}T_0^3. & \end{aligned}$$

Then  $I \subseteq \ker f$  as is easily seen. Thus  $f$  induces an epimorphism

$$\bar{f}: R/I \rightarrow A[Pt, \Delta t^2].$$

First this shows  $\dim R/I \geq 5$ . On the other hand,

$$\dim R/(I, \underline{x}R) = 0, \quad \text{where } \underline{x} = \{T_0, y, z, T_3 - x, S - w\},$$

because of  $R/(I, \underline{x}R) \cong k[x, w, T_1, T_2]/I'$  with

$$\begin{aligned} I' = (wT_1, x^{a-1}wT_2, w^2T_2, x^a, T_1T_3 - wT_2^2, T_1^2, x^{a-1}T_2^2, \\ w^2 - T_1T_2, T_2^3 - xw) \end{aligned}$$

and  $\text{Rad } I' = (x, T_1, w, T_2)$ . Therefore,  $\dim R/I = 5$ . Next we claim that  $R/I$  is a Cohen–Macaulay ring. To this end it is enough to show that  $R_N/IR_N$ , where  $N$  denotes the irrelevant maximal ideal of  $R$  is a Cohen–Macaulay ring (see [12]). Now

$$(R/I)_r, \quad r \in \{y, z, w, T_3, S\} = \underline{y}$$

is a Cohen–Macaulay ring. For  $r \in \{y, z, w\}$  this follows because  $PA_r$  is a

complete intersection in  $A_r$ . For  $r \in \{T_3, S\}$  this follows by some nasty calculations using the above relations.

Since  $\dim R/(I, yR) = 0$ , the non-Cohen–Macaulay locus of  $R_N/IR_N$  is contained in  $V(NR_N/IR_N)$ . That is,  $\underline{x}$  is a reducing system of parameters of  $R_N/IR_N$ . In order to show that  $R_N/IR_N$  is a Cohen–Macaulay ring it is enough to prove that

$$(I, \underline{z}R) : (T_3 - x) = (I, \underline{z}R), \quad \underline{z} = \{T_0, y, z, S - w\}.$$

To this end recall that

$$R/(I, \underline{z}R) \cong k[x, w, T_1, T_2, T_3]/J,$$

where  $J$  denotes  $I$  evaluated at  $\underline{z}R$ . Now

$$\text{Rad } J = (T_1, w, T_2, x) \cap (T_1, w, T_2, T_3).$$

Then it turns out that

$$J : \langle x \rangle = (wT_1, wT_2, T_3, T_1^2, T_2^2, w^2 - T_1T_2, T_2^3 - wT_3)$$

and

$$J : \langle T_3 \rangle = (wT_1, x^{a-1}wT_2, w^2T_2, x^{a-1}, T_1T_3 - wT_2^2, T_1^2, w^2 - T_1T_2, T_2^3 - wT_3).$$

Therefore,  $(J : \langle x \rangle) \cap (J : \langle T_3 \rangle) = J$ , i.e.,  $\underline{x}$  forms an  $R_N/IR_N$ -regular sequence. By the injection

$$R/I \rightarrow (R/I)_z \cong k[x, y, z, 1/z, w, T_0, T_3]/(F_0T_2 - F_2T_0)$$

it follows that  $R/I$  is a five-dimensional Cohen–Macaulay domain. Therefore,  $R/I \cong A[Pt, \Delta t^2]$ . Now  $\text{ht } p(R/I) = 2$ , where  $p = (y, z, w)$ , and  $R/I = S(P)$  as shown in the proof of Proposition 8. Finally by [18, Corollary 3.4], and the modification suggested before Theorem 3.3,  $S(P)$  is a Gorenstein ring.  $\square$

## 5. Concluding remarks

First let us recall the long outstanding problem whether any irreducible, reduced, integral curve  $C$  in  $\mathbb{P}_k^3$  is set-theoretically a complete intersection. If  $\text{char}(k) = 0$  this is not known even for the class of imperfect monomial curves. Let  $P$  denote the defining prime ideal of  $C$  in  $A = k[x, y, z, w]$ . We denote by  $\text{ara } P$  the smallest number of homogeneous elements of  $A$  generating  $P$  up to the radical. In the case  $S(P)$  is an  $A$ -algebra of finite type we have

$$\text{ara } P \leq \dim S(P)/(M) \leq \dim A - 1,$$

where  $M = (x, y, z, w)A$ .

**Corollary 10.** *In both of the cases of Theorem 2 one has  $\dim S(P)/(M) = 3$  and  $P = \text{Rad}(G, H, \Delta)$  respectively  $P = \text{Rad}(F_0, F_2, \Delta)$ .*

**Proof.** The dimensions of  $S(P)/(M)$  follow easily by virtue of the given presentation of  $S(P)$ . A homogeneous system of parameters of  $S(P)/(M)$  yields the desired formulas of  $P$  up to the radical.  $\square$

In contrast to the situation of a one-dimensional prime ideal, the finite type of the symbolic blow-up ring does not imply that  $P$  is set-theoretically a complete intersection. But it is known that a perfect monomial curve in  $\mathbb{P}_k^3$  is a set-theoretic complete intersection (see [13] or [19]). We shall add an argument that  $\text{ara } P = 2$  for the first example considered in Theorem 2. To this end take the syzygy  $wF + y^bG + zH = 0$ , i.e.,

$$y^aG \equiv -zH \pmod{F}$$

and

$$y^{(a+b)b}G^{a+b} \equiv (-1)^{a+b}z^{a+b}H^{a+b} \equiv (-1)^{a+b}x^{a-1}y^bwH^{a+b} \pmod{F}.$$

Because  $y$  is regular modulo  $F$  it follows that

$$y^{(a+b-1)b}G^{a+b} \equiv (-1)^{a+b}x^{a-1}wH^{a+b} \pmod{F}.$$

The sequence  $\{y^{(a+b-1)b}, x^{a-1}w\}$  is regular modulo  $F$ , i.e., there exists a form  $K \in A$  such that

$$y^{(a+b-1)b}K \equiv (-1)^{a+b}H^{a+b} \pmod{F}$$

and

$$x^{a-1}wK \equiv G^{a+b} \pmod{F}.$$

This shows  $K \in P$  and  $P = \text{Rad}(F, K)$ . Using results of [4] a slight modification yields another proof that any perfect monomial curve in  $\mathbb{P}_k^3$  is set-theoretically a complete intersection.

The question whether the second example in Theorem 2 is set-theoretically a complete intersection remains open for  $\text{char}(k) = 0$ .

In order to complete the picture about the question when  $P^n = P^{(n)}$  for all  $n \geq 1$  for the defining prime ideal  $P$  of a monomial curve in  $\mathbb{P}_k^3$ , let us note the following: By Plücker's formula it turns out that  $P_1 = (1:0:0:0)$  and  $P_2 =$

$(0:0:0:1)$  are the only possible singular points of such a curve. Because the local rings at these points correspond to monomial curves in  $\mathbb{A}_k^3$  we have  $P^n A_Q = P^{(n)} A_Q$  for all  $n \geq 1$ ,  $Q = (y, z, w)A$  respectively  $Q = (x, y, z)A$ , if and only if both of the singularities are complete intersection points. Thus a necessary condition for  $P^{(n)} = P^n$  for all  $n \geq 1$  is that the singularities of the curve are at most complete intersections. It is not clear whether this condition is also sufficient. Note that Theorem 1 gives a certain support to an affirmative answer. Moreover, if  $G(P)$  is a Cohen–Macaulay ring respectively the zero ideal of  $G(P)$  is unmixed, then it is true (see [11, Corollary 2.1]). So it would be of some interest to clarify when the form ring  $G(P)$  is a Cohen–Macaulay ring for the prime ideal  $P$  of a monomial curve in  $\mathbb{P}_k^{3,2}$ .

In connection with  $S(P)$ , the symbolic Rees ring, one may consider  $T(P) = \bigoplus_{n \geq 0} P^{(n)}/P^{(n+1)}$ , the symbolic form ring with respect to  $P$ . Note that  $T(P) = S(P)/S(P)_+(1)$ , where  $S(P)_+$  denotes the homogeneous ideal generated by all forms of positive degree. Thus,  $T(P)$  is a Noetherian graded ring, provided  $S(P)$  is a Noetherian graded ring.

**Corollary 11.** *Let  $P$  denote one of the prime ideals considered in Theorem 2. Then  $T(P)$  is a Gorenstein domain.*

**Proof.** By the argument of [9] it follows that  $T = T(P)$  is a domain. Note that  $T \otimes_A A_P \cong G(PA_P)$  is a domain. Set  $S = S(P)$ . Then the short exact sequences

$$0 \rightarrow S_+ \rightarrow S \rightarrow A \rightarrow 0$$

and

$$0 \rightarrow S_+(1) \rightarrow S \rightarrow T \rightarrow 0$$

imply that  $T$  is a graded Cohen–Macaulay ring. Now the canonical module  $K_T$  of  $T$  is given by  $K_T = \text{Ext}_S^1(S/S_+(1), S)$ . Note that  $S$  is a Gorenstein ring with  $\dim S = \dim T + 1$ . For an  $S$ -regular element  $r \in P \setminus P^{(2)}$  it follows

$$K_T \cong \text{Hom}_S(S/S_+(1), S/rS) \cong (rS : S_+(1))/rS \cong S/S_+(1)$$

(see [11, Lemma 1.1]). That is,  $K_T \cong T$  and  $T$  is a Gorenstein ring.  $\square$

There has also been interest in the extended (symbolic) Rees rings

$$R'(P) = R(P)[u], \quad S'(P) = S(P)[u], \quad u = t^{-1},$$

of a prime ideal  $P$  of  $A$ . It is clear that

<sup>2</sup> For further results in this direction see the preprint of S. Huckaba and C. Huneke mentioned in Footnote 1.

$$R'(P)/(u) \cong G(P) \quad \text{and} \quad S'(P)/(u) \cong T(P).$$

In the situation of Theorem 1 and Theorem 2,  $u$  is a prime element of  $R'(P)$  respectively  $S'(P)$ . Because of

$$R'(P)_u \cong A[t, u] \quad \text{respectively} \quad S'(P)_u = A[t, u],$$

it turns out that  $R'(P)$  respectively  $S'(P)$  is a Gorenstein unique factorization domain.

## References

- [1] M. Auslander and D.A. Buchsbaum, Codimension and multiplicity, *Ann. Math.* 68 (1958) 625–657.
- [2] J. Barshay, Graded algebras of powers of ideals generated by  $A$ -sequences, *J. Algebra* 25 (1973) 90–99.
- [3] H. Bresinsky and B. Renschuch, Basisbestimmung Veronesescher Projektionsideale mit allgemeiner Nullstelle  $(t_0^m, t_0^{m-r}t_1^r, t_0^{m-s}t_1^s, t_1^m)$ , *Math. Nachr.* 96 (1980) 257–269.
- [4] H. Bresinsky, P. Schenzel and W. Vogel, On Liaison, arithmetical Buchsbaum curves and monomial curves in  $\mathbb{P}^3$ , *J. Algebra* 86 (1984) 283–301.
- [5] P. Brumatti, A. Simis and W.V. Vasconcelos, Normal Rees algebras, *J. Algebra* 112 (1988) 26–48.
- [6] D. Buchsbaum and D. Eisenbud, Algebra structures for finite free resolutions and some structure theorems for ideals of codimension 3, *Amer. J. Math.* 99 (1977) 447–485.
- [7] D. Eisenbud and C. Huneke, Cohen–Macaulay Rees algebras and their specializations, *J. Algebra* 81 (1983) 202–224.
- [8] S. Eliahou, Symbolic powers of monomial curves, *J. Algebra* 117 (1988) 437–456.
- [9] M. Hochster, Criteria for equality of ordinary and symbolic powers of primes, *Math. Z.* 133 (1973) 53–65.
- [10] S. Huckaba, On complete  $d$ -sequences and the defining ideals of Rees algebras, Preprint, 1988.
- [11] C. Huneke, On the associated graded ring of an ideal, *Illinois J. Math.* 26 (1982) 121–137.
- [12] J. Matijevic and P. Roberts, A conjecture of Nagata on graded Cohen–Macaulay rings, *J. Math. Kyoto Univ.* 14 (1974) 125–128.
- [13] L. Robbiano and G. Valla, Some curves in  $\mathbb{P}^3$  are set-theoretic complete intersections, in: *Algebraic Geometry—Open Problems*, Proc. Ravello 1982, Lecture Notes in Mathematics 997 (Springer, Berlin, 1983) 391–399.
- [14] P. Schenzel, Dualisierende Komplexe in der lokalen Algebra und Buchsbaum-Ringe, *Lecture Notes in Mathematics* 907 (Springer, Berlin, 1982).
- [15] P. Schenzel, Examples of Noetherian symbolic blow-up rings, *Rev. Roumaine Math. Pures Appl.* 33 (1988) 375–383.
- [16] P. Schenzel, Castelnuovo’s index of regularity and reduction numbers. In: *Topics in Algebra*, Banach Center Publications 26 (PWN, Warsaw), to appear.
- [17] P. Schenzel, N.V. Trung and N.T. Cuong, Verallgemeinerte Cohen–Macaulay-Moduln, *Math. Nachr.* 85 (1978) 57–73.
- [18] A. Simis and N.V. Trung, The divisor class group of ordinary and symbolic blow-ups, *Math. Z.* 198 (1988) 479–491.
- [19] J. Stückrad and W. Vogel, On the number of equations defining an algebraic set of zeros in  $n$ -space, *Sem. Eisenbud, Singh, Vogel*, Vol. 2, Teubner-Texte zur Mathematik 48 (Teubner, Leipzig, 1982) 88–107.
- [20] N.V. Trung, Der gradierte Ring bezüglich des Primideals von Macaulay, *Beiträge Algebra Geom.* 11 (1981) 35–39.